



## Static and dynamic aspects of transonicity in Bondi accretion

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**Abstract** Transonicity in a spherically symmetric accreting system has been considered in both the stationary and the dynamic regimes. The stationary flow, set up as a dynamical system, has been shown to be greatly unstable to even the minutest possible deviation in the boundary condition for transonicity. With the help of a simple analytical model, and some numerical modelling, it has then been argued that the flow indeed becomes transonic and stable, when the evolution of the flow is followed through time. The time-dependent approach also shows that there is a remarkable closeness between an equation of motion for a perturbation in the flow, and the metric of an analog acoustic black hole.

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### 1. Introduction

Accretion processes involve, in very simple terms, the flow dynamics of astrophysical matter under the external gravitational influence of a massive astrophysical object, like an ordinary star or a white dwarf or a neutron star [1]. The qualifier “external” is to be stressed upon here, to distinguish accretion processes from the self-gravity driven collapse of a fluid system, as in the case of a star itself. The accreting astrophysical matter whose fluid properties we are interested in, could be the interstellar medium – as modelled by its spherically symmetric infall on to an isolated accretor – or stellar matter, as seen in a binary system, where tidal deformation of a star, leads to matter flowing out from it into the potential well of a compact companion [1]. In all of these cases, the fluid system is satisfactorily described by a momentum balance equation (with gravity as an external force), the continuity equation and a polytropic equation of state.

In astrophysics, studies in accretion, in a formal sense, have been carried out for more than half a century now. Initially, astrophysical problems in the nature of what we understand to be accretion processes at present, were studied by Hoyle and Lyttleton in the context of the infall of matter on to a star moving through the interstellar medium [2]. In their methods, however, Hoyle and Lyttleton neglected the pressure effects, with the argument that any heat generated would be radiated away rapidly, so that the temperature of the infalling gas (and related to that, the effects of pressure as well) would remain negligibly low [2]. This was found to be a satisfactory prescription for most cases of astrophysical interest, which were then being studied. In 1952, however, in a very important paper [2], Bondi attacked this problem somewhat differently, by taking into account the pressure effects. This work on spherically symmetric infall of matter, on to a massive and attracting centre, using formal fluid dynamical equations,

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has assumed a paradigmatic status in accretion studies [1,3]

Bondi studied the problem of spherical accretion in its stationary limit, i.e. by only considering the extreme case of negligible dynamical effects. In his own words the mathematical difficulties associated with the problem of studying both the pressure effects (ignored by Hoyle and Lyttleton) and the dynamical effects were "insuperable" [2], given the computational facilities available to him half a century ago.

The stationary equations would lead to various classes of solutions. Of these the interesting ones would be those which obey the outer boundary condition that at large radii the flow velocity would be highly subsonic, i.e. small compared with the speed of sound, with the speed of sound itself approaching a constant "ambient" value at large radial distances. With this outer boundary condition being satisfied, it would be physically meaningful to consider only the entire class of solutions, which remain subsonic everywhere, and the exceptional case of the lone transonic solution. This exceptional solution, determined uniquely by the value of the density at infinity, crosses the sonic point of the flow – a point which is given by the radius where the flow velocity smoothly matches the speed of sound – and acquires supersonic values at lesser radial distances. Further, in one sense this transonic solution represents a limiting value for the mass inflow rate, for a given value of the density at infinity, because any inflow rate higher than that given by the transonic solution, will make the flow "bounce" outwards [4].

In this situation it became worthwhile to inquire into the natural selection of a solution by a spherically symmetric accreting system. Regarding this question Bondi himself offered some insights when he observed in his paper that since it is with the transonic solution that the lowest energy is associated, we may find a natural system in this state [2]. Furthermore, Bondi suggested that this result would also be 'in agreement with the intuitive idea that, since there is nothing to stop the process of accretion, it takes place at the greatest possible rate' [2], i.e. the rate given by the transonic solution.

These arguments by Bondi were essentially based on physical considerations. His suggestion that a linear stability analysis might offer a clue was also put to the test [4–6]. Subjecting the physically relevant stationary inflow solutions to a linearised time-dependent perturbation led to the result that all solutions displayed stable behaviour,

and through this method, no insight may be had as to the natural inclination of the system for any particular solution. Garlick settled this point quite categorically by stating that "the plausible assumption that subsonic flows are not realised in nature, and that critical flows always develop, cannot be justified by a linear stability analysis, but rather by the more fundamental arguments given by Bondi" [5].

This then was the understanding that became firmly established – that the transonic solution enjoyed primacy over the subsonic ones. It was easy to recognise that for black holes, this would be certainly true. For one thing instead of a physical surface, black holes have what is known as an event horizon, which precludes all possibility of a pressure build-up at smaller radii, for another, all matter reaching the event horizon must do so supersonically implying that it must display transonic behaviour [7].

The situation, however, was not so clear-cut if the accretor had a hard surface like a neutron star or a white dwarf. For such an accretor, it may be supposed that the accumulated matter would build up pressure near the surface and cause the supersonic flow to be shocked down to subsonic levels, although for a neutron star in particular, all accreted matter is expected to be efficiently "vacuum cleaned" away, making it easier for the flow to remain supersonic [4].

Then again there was the conceptual difficulty in understanding the realisability of the transonic solution, by only an infinitely precise determination of the outer boundary condition for the stationary equations of the flow. An infinitesimal error in the determination of the boundary condition would generate a solution far away from transonicity. Within the framework of the stationary picture, it is difficult to imagine that a natural physical system would be so precisely tuned. On the other hand, quite intriguingly, this difficulty is resolved when, instead of merely the stationary equations, the transonic solution is tried to be numerically generated through a time-evolution (accounting for explicit time-dependence) of the spherically symmetric accreting system [8–10].

In this article we make some attempt in addressing these issues. We take up the stationary flow equations first and model them along the lines of the equations governing a dynamical system. This establishes the nature of the sonic point of the flow, as that of a saddle point. Quite apart from the fact that a saddle point is inherently unstable, among other adverse implications, this analysis illustrates why in the stationary picture, after having started

with an outer boundary condition, so much difficulty is encountered in generating a solution that passes through the sonic point [11]. The stationary transonic solution is notoriously unstable under even the smallest of deviations from the infinitely precise boundary condition that would be needed to generate the solution.

Since in a realistic situation, an accreting system would have evolved through only a finite span of time, we then consider the case of explicit time-evolution of the flow solution(s), starting from a reasonably physical initial condition (given at  $t = 0$ ). This gives us to understand quite convincingly that not only would the transonic solution be selected, but that the physical mechanism that is very likely effective in making the selection is the one about which Bondi has conjectured (entirely within the confines of the stationary framework) – that the flow solution with the lowest specific energy associated with it, would be preferred to all the others. It is important to note that this selection mechanism is actually at work through the temporal evolution of the flow, and that it is entirely non-perturbative in character [11].

In keeping with this time-dependent approach to uphold the exclusive selection of the transonic solution from a set of various solutions, we have finally carried out an interesting exercise, that is actually perturbative in nature, by imposing a linearised time-dependent perturbation on the constant matter flow rate, we have been able to identify a remarkable closeness between an equation of motion for the perturbation, and the metric of an acoustic black hole. With the aid of this analogy we have argued that contrary to common belief, even a perturbative treatment might be conveying a subtle hint in favour of transonicity.

## 2. The equations of the spherically symmetric flow

The variables that we need to consider are the velocity (radial velocity only for the spherically symmetric flow),  $v$ , and the density,  $\rho$ . We ignore viscosity and write down the inviscid Euler equation for  $v$  as

$$\frac{dv}{dt} + v \frac{\partial v}{\partial r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} - \frac{\partial V(r)}{\partial r}, \quad (1)$$

where  $P$  is the local pressure and  $V$  is the potential due to the gravity of the central accretor of mass  $M$ , given by  $V(r) = -GM/r$ . The pressure is related to the local density through a polytropic equation of state  $P = K\rho^\gamma$ , in which  $K$  is a constant, and  $\gamma$  is the polytropic exponent. [12],

whose admissible range is given by  $1 < \gamma < 5/3$ , with this range having been restricted by the isothermal and the adiabatic limits, respectively. To know how  $\rho$  evolves, we need the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho v r^2) = 0 \quad (2)$$

Our system is specified by eqs (1) and (2). We are interested in static solutions, the problem of which has been defined in Bondi's own words as follows – “A star of mass  $M$  is at rest in an infinite cloud of gas, which at infinity is also at rest. The motion of the gas is spherically symmetrical and steady, the increase in the mass of the star being ignored so that the field of force is unchanging” [2]. Since transonic flows are our concern, we require that the static flow evolves from  $v \rightarrow 0$  as  $r \rightarrow \infty$  (the outer boundary condition) to  $v > c_s(r)$  for small  $r$ , where  $c_s(r)$  is the speed

of sound given by  $c_s^2 = \partial P / \partial \rho = \gamma K \rho^{\gamma-1}$ .

The stationary solution implies  $\partial v / \partial t = \partial \rho / \partial t = 0$ , and hence we have  $\rho = \rho(r)$  and  $v = v(r)$ . This requirement renders eqs (1) and (2) as

$$v \frac{dv}{dr} + \frac{1}{\rho} \frac{dP}{dr} + \frac{GM}{r^2} = 0 \quad (3)$$

and

$$\frac{1}{v} \frac{dv}{dr} + \frac{1}{\rho} \frac{d\rho}{dr} + \frac{2}{r} = 0, \quad (4)$$

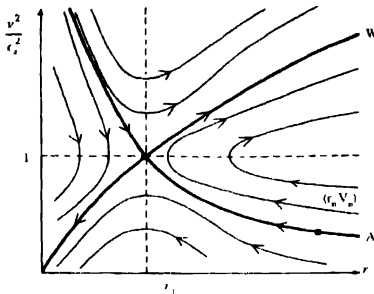
respectively. It is to be noted here that the eqs (3) and (4) remain invariant under the transformation  $v \rightarrow -v$ , i.e. the mathematical problem for inflows ( $v < 0$ ) and outflows ( $v > 0$ ) is the same [13] in the stationary state.

It is in principle possible to eliminate either  $v$  or  $\rho$  and solve for the other variable as a function of  $r$ . However, adopting a slightly different approach, it is possible to recast eqs (3) and (4) in a combined form as

$$\frac{d}{dr} (v^2) = \frac{2v^2}{r} (2c_s^2 - GM/r), \quad (5)$$

in which we use the speed of sound,  $c_s$ , to scale the flow velocity.

The various classes of solutions of eq (5) are shown in Figure 1 (to be presently read without the arrows). The two dark solid curves labelled **A** and **W** refer to the accretion flow and the wind flow, respectively. The meaning of “accretion” and “wind” flows, which emerges from



**Figure 1.** Stationary solutions for spherically symmetric accretion onto a star. The bold solid curves, *A* and *W*, represent "accretion" and "wind", respectively. The fixed point is at  $r = r_0$  and  $(v^2/c_s^2) = 1$ . A linear stability analysis indicates that the fixed point of the flow is a saddle point. The direction of the arrows along the curve *A* demonstrates that the transonic flow is not physically realisable in the stationary framework.

Figure 1 is quite clear. At the point  $v^2 = c_s^2$ , these solutions correspond to a finite slope, i.e. a finite value of  $dv/dr$ . This implies that when  $v^2 = c_s^2$ , we must also have  $2c_s^2 = GM/r$ , implying that the intersection point is a critical point. The "wind" and "accretion" flows smoothly pass through the point  $v^2 = c_s^2 = GM/2r$ , with the sonic length scale having been labelled as  $r = r_0$ .

The question which now arises is that of the natural preference of the accreting system for a particular solution from among all the various possible classes of flows shown in Figure 1. Linear stability analysis of the various stationary inflow solutions in real time, indicates that under the influence of a linearised time-dependent perturbation, all solutions appear to be stable, and thus it offers no clue as to the selection of any particular solution [4–6]. To resolve this issue, it would then be very much worthwhile to recall Bondi's conjecture on this point, that the selection would be in favour of that solution, with which is associated the least total energy. The transonic branch satisfies that criterion, and hence is the choice.

### 3. The spherically symmetric flow as a dynamical system

Thus far we have obtained a seemingly wholesome picture regarding the realisability of the transonic flow. However, upon a closer inspection of Figure 1, we find that there is a problem with it. For solutions which represent flows, the associated sense of direction has been assigned by an arrow to each solution. An integration of  $dv/dr$  would proceed if we start with an initial condition  $v = v_m$  at  $r = r_m$  far away from the star. For a physically realisable

flow, an initial condition infinitesimally close to a point on the accretion line *A*, would trace out a curve infinitesimally close to *A* and in the limit would correctly reproduce *A*, evolving along it and passing through the critical point (sonic point) as we integrate  $dv/dr$ , obtained from Euler's equation. We will soon show that the arrows on the integration route are as shown in Figure 1. It is obvious from the direction of the arrows here that the stationary, spherically symmetric transonic accretion flow is not physically realisable, and closely related to this, the fixed point is also seen to be an unstable saddle point.

To obtain any idea about the direction associated with a solution in the steady picture, it is a matter of common knowledge in the study of dynamical systems [14] that we cannot turn to eq (5) in its present form. Rather, it would be instructive to write eq (5) in a parametrised form

$$\frac{d}{d\tau}(v^2) = 2v^2 \left( 2c_s^2 - \frac{GM}{r} \right) \quad (6)$$

$$\frac{dr}{d\tau} = r(v^2 - c_s^2)$$

As  $\tau$ , which is an arbitrary parameter, evolves ( $\tau$  is not time, since we are dealing with a stationary flow), we generate the  $v(r)$  curves. The particular curves which represent transonic flow are the curves which pass through the fixed point at  $r = r_0$ ,  $v = v_0$ , obtained from eq (5) such that  $v_0^2 = c_{s0}^2$  and  $2c_{s0}^2 = GM/r_0$ . The subscript label 0 represents physical quantities at the critical point. We now need to analyse the nature of the fixed point ( $v_0^2/c_{s0}^2$ ). Writing  $v^2 = v_0^2 + \delta v^2$  and  $r = r_0 + \delta r$ , and linearising in  $\delta v^2$  and  $\delta r$ , we find

$$\frac{d}{d\tau}(\delta v^2) = 2v_0^2 \left[ -(\gamma - 1)\delta v^2 - (2\gamma - 3)\frac{GM}{r_0^2}\delta r \right]$$

$$\frac{d}{d\tau}(\delta r) = r_0 \left[ \frac{\gamma + 1}{2}\delta v^2 + 2(\gamma - 1)\frac{c_{s0}^2}{r_0}\delta r \right] \quad (7)$$

Using solutions of the form  $\delta v^2 = \exp(\lambda\tau)$  and  $\delta r = \exp(\lambda\tau)$ , the eigenvalues of the stability matrix implied by eq (7) are found to be

$$\lambda = \pm c \sqrt{2(5 - 3\gamma)} \quad (8)$$

For the admissible range of  $\gamma$ , i.e.  $1 < \gamma < 5/3$ , the eigenvalues are real with different signs and the fixed point  $(r_0, 1)$  in the  $r - (v^2/c^2)$  space is, therefore, identified as a saddle point. The arrows (characterising a saddle

point) which are needed to make our understanding of the stationary phase portrait complete [14], are as shown in Figure 1. The curves which we have labelled "accretion" and "wind" in Figure 1 are in fact now seen to be the separatrices of a dynamical system, and one cannot traverse the length of a separatrix upto the critical point in any finite range of  $\tau$  values. Starting from an initial point to the right of the fixed point on the curve labelled A, one would need an infinitely large number of steps to reach the fixed point, and will certainly not cross it.

To explicitly establish this result we consider the two parametrised equations in  $\delta v^2$  and  $\delta r$ , given by the eqs (7), and using them, we write

$$\frac{d(\delta v^2)}{d(\delta r)} = \frac{d(\delta v^2)/d\tau}{d(\delta r)/d\tau} \quad (9)$$

We then integrate eq (9) in  $\delta v^2$  and  $\delta r$ , and fix the integration constant from the critical point condition,  $\delta v^2 = \delta \dot{r} = 0$ , to obtain,

$$\delta v^2 = \frac{-2\epsilon_{10}^2}{r_0(\gamma+1)} \left[ 2(\gamma-1) \pm \sqrt{2(5-3\gamma)} \right] \delta r \quad (10)$$

Using eq (10) in the latter of the two relations given by eq (7), we get,

$$\frac{d(\delta r)}{d\tau} = \pm \epsilon_{10}^2 \sqrt{2(5-3\gamma)} \delta r \quad (11)$$

We can integrate eq. (11), for both roots, from an arbitrary initial value of  $\delta r = [\delta r]_m$  to a point  $\delta r = \epsilon$ , where  $\epsilon$  is very close to the critical point given by  $\delta r = 0$ . We thus get

$$\begin{aligned} \tau &= \pm \frac{1}{\epsilon_{10}^2 \sqrt{2(5-3\gamma)}} \int_{[\delta r]_m}^{\epsilon} \frac{d(\delta r)}{\delta r} \\ &= \pm \frac{1}{\epsilon_{10}^2 \sqrt{2(5-3\gamma)}} \ln \left| \frac{\epsilon}{[\delta r]_m} \right|, \end{aligned} \quad (12)$$

from which it is easy for us to see that for  $\epsilon \rightarrow 0$ ,  $|\tau| \rightarrow \infty$ . This implies that the critical point may be reached along either of the separatrices, only after  $|\tau|$  has become infinitely large. That is why the spherically symmetric flow cannot be realised.

It might also be noted from Figure 1 that within the framework of the stationary picture, there is another obstacle of a more practical nature, standing in the way of the realisability of the transonic flow. Each of the inflow solutions – the subsonic ones, the transonic one and the

so called "bouncing solutions" [4], is to be obtained by its own very precisely defined boundary condition. From among the infinitude of possibilities, the boundary condition that would exactly reproduce the transonic accretion curve would have to be defined with infinite precision. And yet, even if that practical difficulty were to be satisfactorily addressed, by dint of the fixed point being a saddle, the transonic inflow solution would still not be generated spontaneously.

To illustrate all these points further we carry out a simple numerical analysis. An integration of eq (3) with the help of the polytropic equation of state, and its relation to the speed of sound, will give

$$\frac{v^2}{2} + n c_s^2 - \frac{GM}{r} = E, \quad (13)$$

in which,  $n$  is the polytropic index, given by  $n = (\gamma - 1)^{-1}$  [12], while the integration constant is fixed as  $E = n c_s^2(\infty)$ , for the boundary condition  $v \rightarrow 0$ ,  $c_s \rightarrow c_s(\infty)$  for  $r \rightarrow \infty$ . Integration of the continuity equation, as given by eq (4), will yield

$$4\pi \rho v r^2 = m, \quad (14)$$

in which the integration constant  $m$  (mass accretion rate) is given by [1]

$$m = \pi G^2 M^2 \frac{\rho_\infty}{c_s^3(\infty)} \left( \frac{2}{5-3\gamma} \right)^{(5-3\gamma)/(2(\gamma-1))} \quad (15)$$

Combining eqs (13) and (14), along with substituting  $\rho$  by its dependence on  $c_s$ , will finally give

$$\frac{v^2}{2} + n \left( \frac{\dot{m}}{v r^2} \right)^{1/n} - \frac{GM}{r} - n c_s^2(\infty) = 0 \quad (16)$$

in which  $\mu = (m/4\pi \rho_\infty) c_s^{2n}(\infty)$

We solve eq (16) for  $v$  numerically by the bisection method, using the values  $M = M_\odot$ ,  $c_s(\infty) = 10 \text{ km s}^{-1}$ ,  $\rho_\infty = 10^{-21} \text{ kg m}^{-3}$  and  $n = 2.5$ . All these values are typical of accretion of the interstellar medium on to an average star. Corresponding to a given boundary condition, each value of  $r$  in eq (16) would give a set of two real and meaningful solutions. The sequence of data points obtained would indicate that the twin solutions would look as shown in Figure 2 (in which the flow velocity has been scaled as the Mach number), which supports our contention about the non-realisability of the critical solutions within the framework set up by the stationary equations alone.

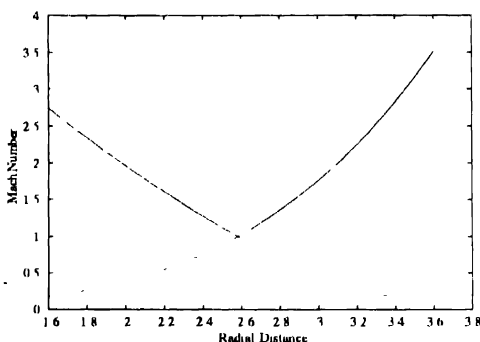


Figure 2. A plot of the Mach number,  $M$ , against the radial distance  $r$  (given on a logarithmic scale). The sequence in which the data points have been numerically generated has been shown here by the two curves.

#### 4. Dynamic selection of separatrices : a model

In the Section 3 we discussed the non-realizability of the stationary transonic solutions if the hydrodynamic accretion problem were to be studied solely in the stationary limit. The non-realizability arises due to the fact that the stationary transonic solutions are actually separatrices of various classes of solutions (as the direction of the arrows in Figure 1 would show) in a dynamical system. However, it is widely maintained that transonic solutions are indeed to be found in a natural system, as has been seen for the case of the solar wind [15,16], which may be treated as a spherically symmetric transonic outflow solution. To reconcile this observational fact with our study of the stationary flow picture, it must be appreciated that a real astrophysical problem is not stationary but dynamic (time-evolutionary) in nature. Therefore we need to take into account explicit time-dependence of the flow variables concerned. In that case it becomes evident that the actual velocity profile would not only depend on the radial distance,  $r$ , but also on time,  $t$ . In this dynamic situation, it is our contention that all the adverse implications regarding transonicity would disappear, and a transonic flow would be realised.

To assure ourselves that such indeed should be the case, we first consider a tractable mathematical model problem. We choose a differential equation given by

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)} = \frac{x + y - 2}{y - x}, \quad (17)$$

whose integral can be written as

$$x^2 - y^2 - 4x + 2xy = -C', \quad (18)$$

with  $C'$  being a constant. This is the equation of a hyperbola. If we want that particular solution which passes through the point where  $f(x, y) = g(x, y) = 0$ , namely  $x = y = 1$ , then  $C' = 2$ . The curve  $x^2 - y^2 - 4x + 2xy = -2$  factorises into a pair of straight lines  $y - x(1 + \sqrt{2}) = 0$  and  $y - x(1 - \sqrt{2}) - \sqrt{2} = 0$ , which are the asymptotes of the hyperbola. This pair is shown in Figure 3 as the lines marked  $A'$  and  $W'$ .

We want to explore the process of drawing the line  $A'$  from a given starting condition. On the line  $A'$ , we have  $y = 0$  at  $x = 2 + \sqrt{2}$ . Let us begin our generating a solution with the condition  $y = 0$  at  $x = 2 + \sqrt{2} - \epsilon$ , where  $0 < \epsilon \ll 1$ . This starting condition fixes the constant  $C'$  as  $C' = 2[1 + \sqrt{2}\epsilon - \epsilon^2/2]$ . Using this value of  $C'$ , we can plot the curve given by eq. (18). For a given value of  $x$ , the value of  $y$  is given by the relevant root of the quadratic equation thus obtained. The two roots are

$$y = x \pm \sqrt{2}[(x-1)^2 + \sqrt{2}\epsilon - \epsilon^2/2]^{1/2} \quad (19)$$

Clearly, to satisfy  $y = 0$  at  $x = 2 + \sqrt{2} - \epsilon$ , the negative sign has to be chosen in eq. (19), which will give

$$y = x - \sqrt{2}[(x-1)^2 + \sqrt{2}\epsilon - \epsilon^2/2]^{1/2} \quad (20)$$

At  $x = 0$ ,  $y = -\sqrt{2}(1 + \sqrt{2}\epsilon - \epsilon^2/2)^{1/2}$ , very different from  $y = \sqrt{2}$ , which one gets on the line  $A'$ . In the limit of  $\epsilon \rightarrow 0$ , one generates a part of  $A'$  ( $x \geq 1$ ) and a part of  $W'$  ( $x \leq 1$ ), instead of the whole of line  $A'$ . Another way of stating this is that the tracing of  $A'$  is utmost sensitive to initial conditions. If we make an error of an infinitesimal amount  $\epsilon$  in prescribing the initial condition on  $A'$ , i.e. if we prescribe  $y = 0$  at  $x = 2 + \sqrt{2} - \epsilon$  instead of  $y = 0$  at  $x = 2 + \sqrt{2}$ , then the "error" made at  $x = 0$  relative to  $A'$  is  $2\sqrt{2}$  which is  $\mathcal{O}(1)$ . An infinitesimal separation at one point leads to a finite separation at a point a short distance away. This is what we mean by saying that the line  $A'$  (and similarly  $W'$ ) should not be physically realised.

The clearest and most direct understanding of the difficulty is achieved by recasting eq. (17) as a first-order autonomous dynamical system, described by the set of differential equations

$$\frac{dy}{d\tau} = x + y - 2$$

$$\frac{dx}{d\tau} = y - x \quad (21)$$

with  $\tau$  being some convenient parametrisation. The fixed

point of this dynamical system is (1,1) namely the point where  $f(x, y)$  and  $g(x, y)$  vanish simultaneously – the point through which  $A'$  and  $W'$  pass. Linear stability analysis of this fixed point in the  $\tau$  parameter space shows that it is a saddle point with the eigenvalues  $\lambda$  given by  $\lambda = \pm \sqrt{2}$ . The solutions passing through the critical point in this  $x$  space can now be drawn with arrows and the result is as shown in Figure 3. The distribution of the arrows

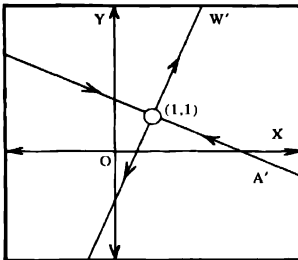


Figure 3. Integration of eq. (17) gives a pair of straight lines, with the integration constant fixed by the intersection point (1,1). In the figure the lines are marked  $A'$  and  $W'$ . Linear stability analysis indicates that on  $A'$  and  $W'$ , the intersection point (1,1) is actually a saddle point, for which the arrows are as shown above.

characterising a saddle point [14] implies  $A'$  and  $W'$  cannot be physically realised.

We now investigate if these apparently non-realisable separatrices in the stationary limit may indeed be realised when we follow the evolutionary dynamics of  $y$  through another variable  $t$ . Accordingly we return to our pedagogic example of eq. (17) but now consider  $y$  as a field  $y(x, t)$  with the evolution through  $t$  represented by

$$\frac{\partial y}{\partial t} + (y - x) \frac{\partial y}{\partial x} = y + x - 2 \quad (22)$$

The stationary solution  $y(x)$  satisfies eq. (17) and the discussion that follows eq. (17) is valid for  $y(x)$  here. The stationary solutions  $y(x)$  are as shown in Figure 3 and the separatrices are  $y(x) = x(1 + \sqrt{2}) - \sqrt{2}$  and  $y(x) = -x(\sqrt{2} - 1) + \sqrt{2}$ . We will now show that the dynamics actually preferentially selects these separatrices.

The general solution of eq. (22) can be obtained by the method of characteristics [17]. The two characteristic solutions of eq. (22) are obtained from

$$\frac{dt}{1} = \frac{dx}{y - x} = \frac{dy}{y + x - 2} \quad (23)$$

They are

$$y^2 - 2xy - x^2 + 4x = C$$

$$x - 1 \pm \frac{1}{\sqrt{2}}(y - x) \left| e^{\pm \sqrt{2}y} \right| = \bar{C} \quad (24)$$

with the latter having been derived by integrating the  $dx/dt$  equation with the help of the solution of the  $dy/dx$  equation. The general solution of these two equations is given by the condition  $C = \zeta(\bar{C})$ , where  $\zeta$  is an arbitrary function, whose behaviour is to be determined by the initial condition at  $t = 0$ .

As in a physically realistic situation, we impose the condition that the evolution is driven through a positive range of values of  $t$  ("time"). This requirement will choose the upper sign in eqs. (24), and the general solution of eq. (22) can then be written as

$$y^2 - 2xy - x^2 + 4x = \zeta \left( \left[ x - 1 + \frac{1}{\sqrt{2}}(y - x) \right] e^{\sqrt{2}y} \right) \quad (25)$$

We give the initial condition that  $y(x) = 0$  at  $t = 0$  for all  $x$ . This leads to

$$\zeta \left[ x \left( 1 - \frac{1}{\sqrt{2}} \right) - 1 \right] = -x^2 + 4x, \quad (26)$$

and gives a form for the function  $\zeta$  as  $\zeta(z) = Az^2 + Bz + C$ , in which,

$$A = \frac{2}{(\sqrt{2} - 1)^2}, B = \frac{4}{\sqrt{2} - 1}, C = 2$$

With the initial condition  $y(x) = 0$  at  $t = 0$ , the solution to eq. (25) reads

$$\begin{aligned} & \left[ y - x(\sqrt{2} + 1) + \sqrt{2} \right] \left[ y + x(\sqrt{2} - 1) - \sqrt{2} \right] \\ & = B\phi e^{\sqrt{2}y} + A\phi^2 e^{-2\sqrt{2}y} \end{aligned} \quad (27)$$

with

$$\phi \equiv \phi(x, y) = x \left( 1 - \frac{1}{\sqrt{2}} \right) - 1 + \frac{y}{\sqrt{2}}$$

Clearly as  $t \rightarrow \infty$ , the right hand side in eq. (27) tends to zero and we approach one of the two separatrices (which were otherwise non-realisable from the stationary viewpoint) shown in Figure 3. Of the two separatrices, the one which will be relevant will be determined by some other imposed requirement. For the astrophysical flow, the two separatrices are the transonic accretion and the wind solutions. One chooses the proper sign of the velocity ( $v < 0$  for inflows,

and  $v > 0$  for outflows) to get the flow in which one is interested

### 5. A non-perturbative selection of the transonic flow

We have seen with the help of a model problem that the dynamics makes it possible as a matter of a mathematical principle, to select solutions which were apparently non-realizable from the stationary perspective. We now extend that treatment to the accretion problem here. It is important to understand that a real astrophysical flow itself is dynamic in character. This implies that explicit time-dependence of the flow equations would have to be taken into account. Having said that, it must also be said that the eqs (1) and (2), which govern the temporal evolution of the flow, do not lend themselves easily to a ready mathematical analysis, indeed, in the matter of incorporating both the dynamical and the pressure effects in the equations, short of a direct numerical treatment, the mathematical problem, in Bondi's own word – “insuperable” [2] – is very appropriately described. Therefore, to have an appreciation of the governing mechanism that underlies any possible selection of a transonic flow, we have to adopt some simplifications.

In our accretion problem we study the dynamics in the regime of what is understood to be the pressureless motion of a fluid in a gravitational field [18] – which is a line of attack that is somewhat reminiscent of the methods of Hoyle and Lyttleton, as Bondi has mentioned in his paper [2]. Simplification of the mathematical equations, however, is not the only justification for such a prescription. A greater justification lies in the fact that the result delivered is in conformity with, what Garlick calls “the more fundamental arguments of Bondi” [5], that it is the criterion of minimum total energy associated with a solution, that will accord it a primacy over all the others.

An immediate consequence of adopting dynamic equations is that the invariance of the stationary solutions under the transformation  $v \rightarrow -v$ , is lost. As a result, we now have to separately consider either the inflows ( $v < 0$ ) or the outflows ( $v > 0$ ), a choice that we impose upon the system at  $t = 0$ . Euler's equation, tailored according to our simplifying requirements, is rendered as

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{GM}{r^2} = 0, \quad (28)$$

which we solve by the method of characteristics [17]. The characteristic curves are obtained from

$$\frac{dt}{1} = \frac{dv}{v} = \frac{-GM}{r^2} \quad (29)$$

On first solving the  $dv/dr$  equation, we get

$$\frac{v^2}{2} - \frac{GM}{r} = \frac{c^2}{2}, \quad (30)$$

with  $c$  being an integration constant obtained from the spatial part of the characteristic equation. We use this result to solve the  $dr/dt$  equation from eq (29), and for  $c^2 > 0$ , we get

$$\frac{2}{cr_i} \left( vr - c^2 t \right) - \ln \left[ \frac{r}{r_i} \left( \frac{v}{c} + 1 \right) \right] = \bar{c} \quad (31)$$

in which  $\bar{c}$  is another integration constant, and  $r_i$  is a length scale in the system defined as  $r_i = 2GM/c^2$ . A similar expression can also be written for  $c^2 < 0$ .

A general solution of eq. (29) is given by the condition,  $\bar{c} = \xi(c^2/2)$ , with  $\xi$  being an arbitrary function, whose form is to be determined from the initial condition. We therefore, set down the general solution as

$$\frac{2}{cr_i} \left( vr - c^2 t \right) - \ln \left[ \frac{r}{r_i} \left( \frac{v}{c} + 1 \right) \right] = \xi \left( \frac{v^2}{2} - \frac{GM}{r} \right) \quad (32)$$

to determine whose particular form we use the initial condition,  $v = u_0(r)$  at  $t = 0$  for all  $r$ , where  $u_0$  is in general some initial velocity distribution over space. It should be easy to see that for  $t \rightarrow \infty$ , we would get the stationary solution

$$\frac{v^2}{2} - \frac{GM}{r} = 0, \quad (33)$$

with the long-time evolutionary approach towards this stationary state behaving as  $t^{-2/3}$ .

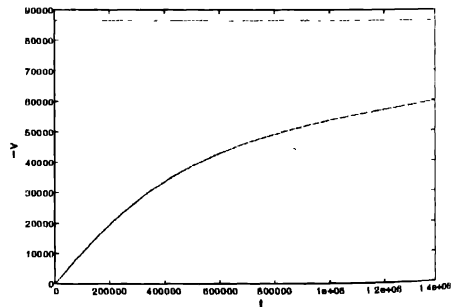


Figure 4 Evolution of the velocity field as given by eq. (29), under the initial condition  $v = 0$  at  $t = 0$  for all  $r$ . The horizontal line on top of the plot represents a limiting value for the velocity,  $\sqrt{2GM/r_i}$ , which is being terminally approached by  $-v$ , whose evolution through  $t$  is being followed at the fixed length scale,  $r = 51r_0$ , with  $r_0$  being the radius of the accretor and with  $M = M_\odot$ .



For a simple intuitive understanding of the physical criterion that drives the flow towards a chosen stationary end, we set  $u_0 = 0$ . This initial condition will necessitate  $c^2 < 0$ . The whole physical picture could be conceived of as one in which a system with a uniform velocity distribution  $v = 0$  everywhere, suddenly has a gravity mechanism switched on in its midst at  $t = 0$ . This induces a potential  $-GM/r$  at all points in space. The system then starts evolving to restore itself to another stationary state, so that for  $t \rightarrow \infty$ , the total energy at all points,  $E = (v^2/2) - (GM/r) = 0$ , remains the same as at  $t = 0$ . This is evidently the stationary solution associated with the lowest possible total energy, and the temporal evolution selects this particular solution from all possible meaningful solutions.

This result has been borne out by a numerical integration of eq. (29) by the finite differencing technique. The mass of the accretor has been chosen to be  $M_0$ , while its radius is  $r_0$ . The evolution through time has been followed at a fixed length scale of  $51r_0$ . The result of the numerical evolution of the velocity field,  $-v$  (for inflows  $v$  is actually negative), through time,  $t$ , has been plotted in Figure 4. The limiting value of the velocity, as the evolution progresses towards the long-time limit, is evidently  $\sqrt{2GM/r}$ , (with  $M = M_0$  and  $r = 51r_0$ ), as eq. (33) would give us to believe. This is what the plot in Figure 4 shows, as  $-v$  approaches its terminal value for  $t \rightarrow \infty$ . The slope of the logarithmic plot of  $-v$  against  $t$  in Figure 5 indicates that in the early stages of the evolution there is a linear growth of the velocity field through time, but on later times, conspicuous deviation from linearity sets in.

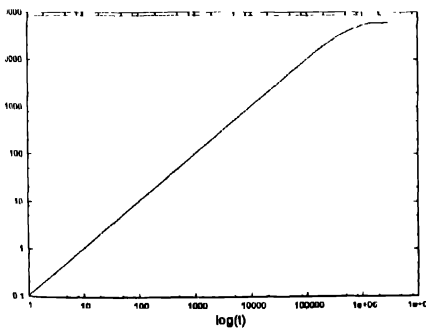


Figure 5 The slope of this logarithmic plot shows that in the early stages of the evolution,  $-v$  varies linearly with  $t$ . Deviation from this linear growth sets in on time scales of  $10^5$  seconds. The horizontal line on top is the limiting value of the velocity

The foregoing argument can now be extended to understand the dynamic selection of the transonic solution. The inclusion of the pressure term in the dynamic equation, fixes the total energy of the system accordingly at  $t = 0$ . A physically realistic initial condition should be that  $v = 0$  at  $t = 0$ , for all  $r$ , while  $\rho$  has some uniform value. The temporal evolution of the accreting system would then non-perturbatively select the transonic trajectory, as it is this solution with which is associated the least possible energy configuration. This argument is in conformity with Bondi's assertion that it is the criterion of minimum total energy that should make a particular solution (the transonic solution in this case), preferred to all the others. However, this selection mechanism is effective only through the temporal evolution of the flow.

To test this contention a numerical study has been carried out, once again using finite differencing, but this time using both the dynamic equations for the velocity and the density fields, as given by eqs. (1) and (2). The accretor has been chosen to have a mass,  $M_0$ , and radius,  $r_0$ . The "ambient" conditions are  $c_s(\infty) = 10 \text{ km s}^{-1}$  and  $\rho_\infty = 10^{-21} \text{ kg m}^{-3}$ , while the polytropic index,  $n = 1.6$ . For these values of the physical constants, transonicity becomes apparent even at the very early stages of the evolution. The course of the evolution of the velocity field (scaled by the speed of sound), at various points of time, for a substantially representative range of the radial distance (scaled by the sonic radius) has been shown in Figure 6.

The outward propagation of the sonic front, as time

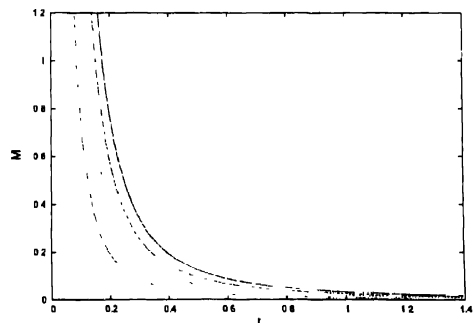
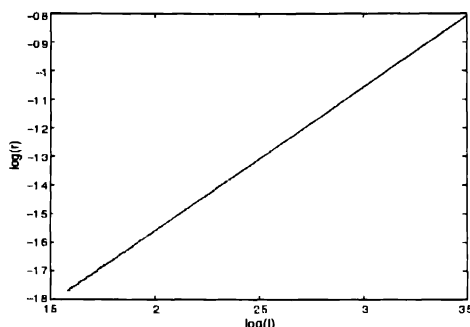


Figure 6. Evolution of the velocity field, scaled as the Mach number,  $M$ , through time,  $t$ . Transonicity is clearly evident, as all curves cross the  $M = 1$  line. Moving from left to right, successive solutions have been shown for  $t = 1000, 2000, 3000$  and  $4000$  seconds, respectively. The radial distance along the horizontal axis has been scaled by the sonic radius,  $r_0$ .

progresses, has been traced in Figure 7. In this logarithmic plot, what can be seen in the early stages of the evolution, is that the sonic front travels through space with a  $1/2$  power dependence on time. This behaviour can be set



**Figure 7** Progression of the sonic front in the early stages of the temporal evolution of the flow. Once again the radial distance along the horizontal axis has been scaled by the sonic radius,  $r_0$ . The slope indicates a power dependence of  $1/2$  for  $r$  versus  $t$ .

down as  $r = r_0 Q t^{1/2}$ , with the constant factor  $Q$  having been determined empirically from dimensional considerations as  $Q = 4(5 - 3\gamma)^{-1} \sqrt{c_s^3(\infty)/GM}$ . This estimate tallies very closely with the numerical value obtained from the plot in Figure 7, notwithstanding which, a cautionary note that has to be sounded here is that this quantitative match holds good for the early stages of the evolution only, and need not be strictly applicable for the entire span of the temporal evolution of the velocity field.

## 6. A perturbation equation and the metric of an acoustic black hole

We have discussed many times in the previous sections that subjecting the stationary solutions to a linearised time-dependent perturbation shows that all the acceptable solutions are stable. Therefore, through a perturbative analysis, no clue could be had as to the special status of any solution. We subject this line of thinking to a closer inspection.

To carry out a linear stability analysis, Petterson *et al* [4] and Theuns and David [6] have made use of a variable defined as  $f = \rho v^2$ , whose stationary value is to be obtained from the continuity equation, given by eq. (4), and this background value,  $f_b$ , is seen to be a constant that is physically identified with the mass flux. In spherical symmetry, the flow variables are  $v$  and  $\rho$ . If we impose small perturbations,  $v'$  and  $\rho'$ , on the stationary background

solutions,  $v_b$  and  $\rho_b$ , we may derive a linearised relation for the perturbation of  $f$  as  $f' = (v' \rho_b + v_b \rho') r^2$ , and in terms of this perturbed quantity we may then obtain a linearised equation of motion for the perturbation, given by

$$\frac{\partial}{\partial t} \left[ \frac{v_b}{f_b} \left( \frac{\partial f'}{\partial t} \right) \right] + \frac{\partial}{\partial r} \left[ \frac{v_b}{f_b} \left( \frac{\partial f'}{\partial r} \right) \right] + \frac{\partial}{\partial r} \left[ \frac{v_b^2}{f_b} \left( \frac{\partial f'}{\partial t} \right) \right] + \frac{\partial}{\partial r} \left[ \frac{v_b}{f_b} (v_b^2 - c_{sb}^2) \frac{\partial f'}{\partial r} \right] = 0, \quad (34)$$

with  $c_{sb}$  being the background stationary value of the speed of sound.

We now approach this whole question from a different perspective. It is known that there is a close one-to-one correspondence between certain features of black hole physics and the physics of supersonic acoustic flows [19]. For an irrotational, inviscid and barotropic fluid flow, Euler's equation may be written as

$$\frac{dv}{dt} + \frac{1}{2} \nabla (v \cdot v) + \frac{\nabla P}{\rho} + \nabla V = 0, \quad (35)$$

in which  $V = -GM/r$ , as before. In this situation we can represent velocity as the gradient of a scalar function  $\psi$ , i.e.  $v = -\nabla \psi$ . For the barotropic condition  $\rho = \rho(P)$ , it is possible to write  $\nabla h = (\nabla P)/\rho$ , upon which, from eq. (35) we may derive the result

$$-\frac{\partial \psi}{\partial t} + \frac{1}{2} (\nabla \psi)^2 + h + V = 0 \quad (36)$$

On a time-dependent background solution  $(\rho_b, P_b, \psi_b)$  we impose a perturbation  $(\rho', P', \psi')$ , and together with eq. (36), and the continuity equation given by eq. (2), a linear equation for the perturbation is delivered as

$$\frac{\partial}{\partial t} \left[ \frac{\rho_b}{c_{sb}^2} \left( \frac{\partial \psi'}{\partial t} + v_b \cdot \nabla \psi' \right) \right] + \nabla \cdot \left[ -\rho_b \nabla \psi' + v_b \frac{\rho_b}{c_{sb}^2} \left( \frac{\partial \psi'}{\partial t} + v_b \cdot \nabla \psi' \right) \right] = 0, \quad (37)$$

in which  $c_{sb}^2 = P'/\rho'$ . This result represents an equation of motion of an acoustic disturbance. Visser discusses this subject [19] by stating that if a "fluid is barotropic and inviscid, and the flow is irrotational (though possibly time-dependent) then the equation of motion for the velocity potential describing an acoustic disturbance is identical to the d'Alembertian equation of motion for a minimally

coupled massless scalar field propagating in a (3 + 1)-dimensional Lorentzian geometry". In other words, eq (37) may alternatively be represented by a compact formulation as,

$$\frac{\partial}{\partial x^\mu} \left( g^{\mu\nu} \frac{\partial}{\partial x^\nu} \psi' \right) = 0, \quad (38)$$

in which the Greek indices run from 0 to 3. The two expressions given by eqs (37) and (38) are completely equivalent [19]. If we consider the special case of only radial dependence for  $g^{\mu\nu}$ , then identifying  $t$  as 0 and  $r$  as 1, the Greek indices, we will get from eq (38),

$$g^{00} = 1, \quad g^{01} = g^{10} = v_b, \quad g^{11} = v_b^2 - c_s^2, \quad (39)$$

Interestingly enough, the expression for  $f'$ , as eq (34) shows it, can also be reduced to a similarly compact form. Eq (38) gives for  $\psi'$ , with an equivalent identification of an effective metric  $g^{\mu\nu}$ . It is easy to compare terms between eqs (34) and (38) with  $\psi'$  replaced by  $f'$ , and see if the same set of  $g^{\mu\nu}$  is obtained for  $f'$ , as eq (39) gives for  $\psi'$ . This closeness of form is very intriguing. In physics of supersonic acoustic flows closely depends to many features of black hole physics. For black hole, infalling matter crosses the event horizon centrally, i.e. at the greatest possible speed. By analogy the same thing may be said of matter crossing the sonic horizon of a spherically symmetric fluid flow, falling on to a point sink. That this fact can be appreciated for the spherically symmetric accretion problem, through a comparative result, as given by eq (34), is quite remarkable. This is because conventional wisdom tells us that we could be quite unable to have any understanding of the overall status of any inflow solution solely through a perturbative technique [4,5]. It is the transonic solution that crosses the sonic horizon at the greatest possible speed and the similarity of the form between eqs (34) and (37) may very well be indicative of the primacy of the transonic solution. If such an insight were truly to be had with the help of the perturbation equation, then the perturbative linear stability analysis might not have been carried out in vain after all.

## 7. Concluding remarks

We have seen how achieving transonicity becomes distinctly easy when one accounts explicitly for time-dependence in the mathematical problem of spherically symmetric accretion. Certainly the physical and mathematical difficulties associated with a purely static approach to this

question, disappear immediately upon involving time. But that is not to say that all questions have been answered satisfactorily. For instance, a very important issue that has to be addressed in greater detail is the manner in which the temporal evolution drives the velocity field towards its stationary critical (i.e. transonic) end, especially in the long-time limit. Even in the simplified pressure-free regime, we have seen for ourselves that this is not something whose answer may be given very easily. With the involvement of the evolution of both the velocity and the density fields – as it has to be for a real astrophysical problem – the computational difficulties will be quite staggering. Nevertheless, this has to be the subject of a more intensive scrutiny.

A further intriguing issue is whether or not a perturbative analysis in real time – considered of not much help in understanding the primacy of the transonic state – can indeed offer some insight into questions related to transonicity. The closeness of an equation of motion for a perturbation in the accretion problem, to the metric of an acoustic black hole, has been a beguiling revelation in this regard.

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